

General solutions for some classes of interacting two field kinks

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February 1, 2008

Abstract

In this work we present some classes of models whose the corresponding two coupled first-order nonlinear equations can be put into a linear form, and consequently be solved completely. In these cases the so called trial orbit method is completely unnecessary. We recall that some physically important models as, for instance, the problem of tiling a plane with a network of defects and polymer properties are in this class of models.

PACS numbers: 11.27.+d, 11.30.Er

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A rapid look at the history of physics is enough to lead anyone to conclude that, fortunately, the most part of the natural physical systems can be studied by using linear differential equations, with their good properties like the superposition principle. Notwithstanding, there are some classes of important systems which are intrinsically nonlinear and, nowadays, there is a growing interest in dealing with such systems [1] - [16]. Unfortunately, as a consequence of the nonlinearity, in general we lose the capability of getting the complete solutions. In this work we show that for those systems in 1+1 dimensions, whose second-order differential equations can be reduced to the solution of corresponding first-order equations, the so called Bogomol'nyi-Prasad-Sommerfield (BPS) topological solitons [17], one can obtain a differential equation relating the two coupled fields which, once solved, leads to the general orbit connecting the vacua of the model. In fact, the "trial and error" methods historically arose as a consequence of the intrinsic difficulty of getting general methods of solution for nonlinear differential equations. About two decades ago, Rajaraman [18] introduced an approach of this nature for the treatment of coupled relativistic scalar field theories in 1+1 dimensions. His procedure was model independent and could be used for the search of solutions in arbitrary coupled scalar models in 1+1 dimensions. However, the method is limited in terms of the generality of the solutions obtained and is convenient and profitable only for some particular, but important, cases [19]. Some years later, Bazeia and collaborators [20] applied the approach developed by Rajaraman to special cases where the solution of the nonlinear second-order differential equations are equivalent to the solution of corresponding first-order nonlinear coupled differential equations. By the way, Bazeia and collaborators wisely applied their solution to a variety of natural systems, since polymers up to domain walls. In this work we are going to present a procedure which is absolutely general when applied to systems like those described in [20], namely the BPS topological solutions. Furthermore, we are going also to show that many of the systems studied in [20]-[25] can be mapped into a first-order linear differential equation and, as a consequence, can be solved in order to get the general solution of the system. After that, we trace some comments about the consequences coming from these general solutions.

In order to deal with the problem, following the usual procedure to get BPS [17] solutions for nonlinear systems, one can particularize the form of

the Lagrangian density

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 - V(\phi, \chi), \quad (1)$$

by imposing that the potential must be written in terms of a superpotential like

$$V(\phi, \chi) = \frac{1}{2} \left(\frac{\partial W(\phi, \chi)}{\partial \phi} \right)^2 + \frac{1}{2} \left(\frac{\partial W(\phi, \chi)}{\partial \chi} \right)^2. \quad (2)$$

The energy of the so-called BPS states can be calculated straightforwardly, giving

$$E_B = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[\left(\frac{d\phi}{dx} \right)^2 + \left(\frac{d\chi}{dx} \right)^2 + W_\phi^2 + W_\chi^2 \right], \quad (3)$$

which lead us to

$$E_B = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[\left(\frac{d\phi}{dx} - W_\phi \right)^2 + \left(\frac{d\chi}{dx} - W_\chi \right)^2 + 2 \left(W_\chi \frac{d\chi}{dx} + W_\phi \frac{d\phi}{dx} \right) \right], \quad (4)$$

and finally to

$$E_B = |W(\phi_j, \chi_j) - W(\phi_i, \chi_i)|, \quad (5)$$

where ϕ_i and χ_i are the i -th vacuum state of the model [22].

In this case, one can easily see that solutions with minimal energy of the second-order differential equation for the static solutions in 1+1 dimensions, can be solved through the corresponding first-order coupled nonlinear equations

$$\frac{d\phi}{dx} = W_\phi(\phi, \chi), \quad \frac{d\chi}{dx} = W_\chi(\phi, \chi), \quad (6)$$

where $W_\phi \equiv \frac{\partial W}{\partial \phi}$ and $W_\chi \equiv \frac{\partial W}{\partial \chi}$. Here, it is important to remark that the BPS solutions settle into vacuum states asymptotically. In other words, the vacuum states act as implicit boundary conditions of the BPS equations.

Now, instead of applying the usual trial-orbit approach [20]-[25], we note that it is possible to write the following equation

$$\frac{d\phi}{W_\phi} = dx = \frac{d\chi}{W_\chi}, \quad (7)$$

where the spatial differential element is a kind of invariant. So, one obtains that

$$\frac{d\phi}{d\chi} = \frac{W_\phi}{W_\chi}. \quad (8)$$

This last equation is, in general, a nonlinear differential equation relating the scalar fields of the model. Now, if one is able to solve it completely, the function $\phi(\chi)$ can be used to eliminate one of the fields, so rendering the equations (6) uncoupled. Finally, this uncoupled first-order nonlinear equation can be solved in general, even if numerically.

From now on, we choose a particular model which can be used for modelling a number of systems [22], in order to exemplify the method in a concrete situation. In fact we will show that for this situation, the equation (8) can be mapped into a linear differential equation, from which it is possible to obtain the general solution. In this case the superpotential is written as

$$W(\phi, \chi) = -\lambda \phi + \frac{\lambda}{3} \phi^3 + \mu \phi \chi^2, \quad (9)$$

and the equation (8) looks like

$$\frac{d\phi}{d\chi} = \frac{\lambda(\phi^2 - 1) + \mu \chi^2}{2\mu \phi \chi}. \quad (10)$$

At this point one can verify that, performing the transformation $\phi^2 = \rho + 1$. The above equation can be written as

$$\frac{d\rho}{d\chi} - \frac{\lambda}{\mu \chi} \rho = \chi, \quad (11)$$

a typical inhomogeneous linear differential equation. It is interesting to observe that its particular solution, corresponds to the result usually presented in the literature [22]. The general solution is easily obtained, giving

$$\rho(\chi) = \phi^2 - 1 = c_0 \chi^{\frac{\lambda}{\mu}} - \frac{\mu}{\lambda - 2\mu} \chi^2, \quad (12)$$

for $\lambda \neq 2\mu$, and

$$\phi^2 - 1 = \chi^2 [\ln(\chi) + c_1], \quad (13)$$

for the $\lambda = 2\mu$ case, and c_0 and c_1 are arbitrary integration constants. It is interesting to note the this last particular situation was not taken into

consideration in the literature up to our knowledge. From now on, we substitute these solutions in one of the equations (6), and solve it, so obtaining a generalized solution for the system. In general it is not possible to solve χ in terms of ϕ from the above solutions, but the contrary is always granted. Here we will substitute $\phi(\chi)$ in the equation for the field χ , obtaining

$$\frac{d\chi}{dx} = \pm 2\mu\chi \sqrt{1 + c_0 \chi^{\frac{\lambda}{\mu}} - \left(\frac{\mu}{\lambda - 2\mu}\right) \chi^2}, \quad (\lambda \neq 2\mu), \quad (14)$$

and

$$\frac{d\chi}{dx} = \pm 2\mu\chi \sqrt{1 + \chi^2 [\ln(\chi) + c_1]}, \quad (\lambda = 2\mu). \quad (15)$$

In general we can not have an explicit solution for the above equations. However one can verify numerically that the solutions are always of the same classes. Notwithstanding, some classes of solutions can be written in closed explicit forms. First of all, we should treat the system when $c_0 = 0$, because in this situation we can solve analytically the system for any value of λ , apart from the case $\lambda = 2\mu$. In this situation we get

$$\chi_+(x) = \frac{2e^{2\mu(x-x_0)}}{1 - ce^{4\mu(x-x_0)}}, \quad \chi_-(x) = \frac{2e^{4\mu(x-x_0)}}{c - e^{4\mu(x-x_0)}}, \quad (16)$$

with $c \equiv -\frac{\mu}{\lambda-2\mu}$. For this choice of the parameters, the solution always vanishes at the boundary ($x \rightarrow \pm\infty$). As a consequence, the corresponding kink solution for the field ϕ , will be given by

$$\phi_+(x) = \pm \frac{ce^{4\mu(x-x_0)} + 1}{ce^{4\mu(x-x_0)} - 1}, \quad \phi_-(x) = \pm \frac{c + e^{4\mu(x-x_0)}}{c - e^{4\mu(x-x_0)}}, \quad (17)$$

which are essentially equivalent to those solutions appearing in [22], given in terms of $\tanh(x)$. Let us now discuss below two particular cases ($c_0 \neq 0$) where the integration can be performed analytically up to the end. Let us first consider the case were $\lambda = \mu$, which has as solutions

$$\chi_+(x) = \frac{4e^{2\mu(x-x_0)}}{[c_0 e^{2\mu(x-x_0)} - 1]^2 - 4e^{4\mu(x-x_0)}}; \quad \chi_-(x) = \frac{4e^{2\mu(x-x_0)}}{[e^{2\mu(x-x_0)} - c_0]^2 - 4}; \quad (18)$$

where we must impose that $c_0 \leq -2$ in both solutions, in order to avoid singularities of the field as can be easily verified. Furthermore, both solutions

vanishes when $x \rightarrow \pm\infty$, provided that $c_0 \neq -2$. On the other hand the corresponding solutions for the field $\phi(x)$ are given by

$$\phi_+(x) = \frac{(c_0^2 - 4) e^{4\mu(x-x_0)} - 1}{[c_0 e^{2\mu(x-x_0)} - 1]^2 - 4 e^{4\mu(x-x_0)}}; \phi_-(x) = \frac{4 - c_0^2 + e^{4\mu(x-x_0)}}{[e^{2\mu(x-x_0)} - c_0]^2 - 4}. \quad (19)$$

Here the first bonus coming from the complete exact solution of the equation (6) comes when we deal with the special case with $c_0 = -2$. It is remarkable that for this precise value of the arbitrary integration constant, an absolutely unexpected kink solution do appears. In fact, it could never be obtained from the usually used solution, where $c_0 = 0$ necessarily. In this special solution, the field χ is a kink with the following asymptotic limits: $\chi_+(-\infty) = 0$ and $\chi_+(\infty) = 1$, and $\phi_+(-\infty) = -1$ and $\phi_+(\infty) = 0$, and correspondingly $\chi_-(-\infty) = 1$ and $\chi_-(\infty) = 0$, and $\phi_-(-\infty) = 0$ and $\phi_-(\infty) = 1$, as it can be seen from an example of a typical profile of this kink in the Fig. 1. Below we present a plot of this kink, which we are going to call type B kink, in contrast with the other cases where the field χ does not have a kink profile, which we call type A kink (see Fig. 2). An interesting observation is that the choice $c_0 = -2$, is precisely the one which makes the right hand side of equation (14) simply proportional to $\chi|1 - \chi| = \zeta \chi(1 - \chi)$, where ζ is the sign function defined as $\zeta \equiv (1 - \chi)/|1 - \chi|$. It takes values ± 1 with $\zeta = +1$ being selected by boundary conditions $0 \leq \chi \leq 1$ for the solutions appearing in (18) and, in this situation, the equation is much easier to solve. In fact, by performing the translation $\chi = \beta + \frac{1}{2}$, we recover a BPS superpotential for the “ $\lambda\phi^4$ ” model, $-(\beta^2 - 1/4)$. A similar situation will happens with the next example.

As the third particular case, we consider the situation where $\lambda = 4\mu$. Now, the exact solutions look like

$$\begin{aligned} \chi_+(x) &= -\frac{2 e^{2\mu(x-x_0)}}{\sqrt{\left[\frac{1}{2} e^{4\mu(x-x_0)} + 1\right]^2 - 4 c_0 e^{8\mu(x-x_0)}}}; \\ \chi_-(x) &= -\frac{4 e^{2\mu(x-x_0)}}{\sqrt{[1 + 2 e^{4\mu(x-x_0)}]^2 - 16 c_0}}; \end{aligned} \quad (20)$$

which have the same asymptotic behavior as that presented in the previous cases for the type A kinks. In other words, provided that $c_0 \neq 1/16$, only

the field ϕ will be a kink. Afterwards, as in the previous case, if one wish to avoid intermediary singularities, one must impose that $c_0 \leq \frac{1}{16}$. Now, the ϕ solutions will be written as

$$\begin{aligned}\phi_+(x) &= \frac{4 + (16c_0 - 1)e^{8\mu(x-x_0)}}{[2 + e^{4\mu(x-x_0)}]^2 - 16c_0 e^{8\mu(x-x_0)}}; \\ \phi_-(x) &= \frac{16c_0 + 4e^{8\mu(x-x_0)} - 1}{[1 + 2e^{4\mu(x-x_0)}]^2 - 16c_0}.\end{aligned}\tag{21}$$

Once more, the particular choice of the integration parameter $c_0 = \frac{1}{16}$, generates a type B kink, with the asymptotic behavior given by: $\chi_+(-\infty) = 0$ and $\chi_+(\infty) = -2$, and $\phi_+(-\infty) = 1$ and $\phi_+(\infty) = 0$, and correspondingly $\chi_-(-\infty) = -2$ and $\chi_-(\infty) = 0$, and $\phi_-(-\infty) = 0$ and $\phi_-(\infty) = 1$.

It is interesting to calculate the energy of these two species of solitonic configurations. For this we use the superpotential (9) and substitute it in the equation (5), and observe that the type A kinks have an energy given by $E_A^{BPS} = \frac{4}{3}\lambda$ and in the two cases considered above ($\lambda = \mu$ and $\lambda = 4\mu$) we obtain $E_B^{BPS} = \frac{2}{3}\lambda$. One could interpret these solutions as representing two kinds of torsion in a chain, represented through an orthogonal set of coordinates ϕ and χ . So that, in the plane (ϕ, χ) , the type A kink corresponds to a complete torsion going from $(-1, 0)$ to $(0, 0)$ and the type B corresponds to a half torsion, where the system goes from $(-1, 0)$ to $(0, 1)$, in the case where $(\lambda = \mu)$ for instance.

In what follows, we will study a more general model, contemplating a number of particular cases which have been studied in the literature, including the previous and some other new ones. For this, we begin by defining the superpotential

$$W(\phi, \chi) = \frac{\mu}{2} \phi^N \chi^2 + G(\phi),\tag{22}$$

which lead us to the following set of equations:

$$\frac{d\phi}{dx} = \frac{dG(\phi)}{d\phi} + \frac{\mu}{2} N \phi^{(N-1)} \chi^2; \quad \frac{d\chi}{dx} = \mu \phi^N \chi.\tag{23}$$

So, the corresponding equation for the dependence of the field ϕ as a function of the field χ , is given by

$$\frac{d\phi}{d\chi} = \frac{\frac{dG(\phi)}{d\phi} + \frac{\mu}{2} N \phi^{(N-1)} \chi^2}{\mu \phi^N \chi}.\tag{24}$$

Now, performing the transformation $\sigma \equiv \phi^2$ we get

$$\frac{d\sigma}{d\chi} = N \chi + \frac{2 G_\phi(\sigma)}{\mu \sigma^{\frac{N-1}{2}}} \frac{1}{\chi}, \quad (25)$$

where $G_\phi(\sigma) \equiv \frac{dG(\phi)}{d\phi} \big|_{\sigma=\phi^2}$. Obviously, there are no arbitrary solutions for the above equation, but for that ones with exact solution we can get the corresponding exact two-field solitons. For instance, let us treat the special case where

$$G_\phi(\sigma) \equiv \frac{dG(\phi)}{d\phi} \big|_{\sigma=\phi^2} = \frac{2(a_0 + a_1 \sigma + a_2 \sigma^2)}{\mu} \sigma^{\frac{N-1}{2}}. \quad (26)$$

The solution will be given by a combination of Bessel functions which, once substituted in the equation for the field χ , lead us to a hardly exactly solvable equation, beyond some singularities which appear in the solution. So, we still here continue to work with the simpler linear case of this equation, where $a_2 = 0$, which furthermore permits us to write arbitrary solutions given by

$$\sigma(\chi) = -\frac{a_0}{a_1} - \frac{N \mu \chi^2}{2(\mu - a_1)} + c_I \chi^{\frac{2a_1}{\mu}}, \quad (27)$$

with $G(\phi)$ given by

$$G(\phi) = \frac{\mu \phi^N}{2} \left[\frac{a_0}{N} + \frac{a_1}{(N+2)} \phi^2 \right], \quad (28)$$

leaving us with the following potential

$$\begin{aligned} V(\phi, \chi) &= \frac{1}{2} \phi^{2(N-1)} a_1^2 \phi^4 + 2 a_1 N \mu \phi^2 \chi^2 + a_0^2 + \mu^2 \chi^2 (4 \phi^2 + N^2 \chi^2) + \\ &+ 2 a_0 (a_1 \phi^2 + N \mu \chi^2), \end{aligned} \quad (29)$$

with c_I being the integration arbitrary constant, and a_0 and a_1 are constants which characterize the physical system. From above, it is easy to conclude that

$$\phi = \pm \sqrt{-\frac{a_0}{a_1} - \frac{N \mu \chi^2}{2(\mu - a_1)} + c_I \chi^{\frac{2a_1}{\mu}}}, \quad (30)$$

and, consequently we are left to solve the following equation

$$\frac{d\chi}{dx} = \pm \mu \left[-\frac{a_0}{a_1} - \frac{N \mu \chi^2}{2(\mu - a_1)} + c_I \chi^{\frac{2a_1}{\mu}} \right]^{\frac{N}{2}} \chi. \quad (31)$$

At this point it is important to remark that many models appearing in the literature can be cast as particular cases from the above general one. For instance if we take $N = 1$, we recover the models I, II and III of [22], and model I of [20]. The case where $N = 2$ is equivalent to the model II in [20] and the model considered in [24].

As a final comment we should say that one can even make a bit generalization of the above exactly solved two fields models. This could be done by starting with the superpotential

$$W_{NM}(\phi, \chi) \equiv G(\phi) + \frac{\mu}{M} \phi^N \chi^M, \quad (32)$$

with $G(\phi)$ being the same appearing previously in the text. After manipulations similar to that one done above, we end with the equation

$$\frac{d\sigma(\chi)}{d\chi} = \left(\frac{2a_1}{\mu}\right) \sigma(\chi) \chi^{(1-M)} + \left(\frac{2N}{M}\right) \chi, \quad (33)$$

where $\sigma \equiv \phi^2 + \left(\frac{a_0}{a_1}\right)$. Solving the above equation for arbitrary M , one obtains that

$$\begin{aligned} \sigma_M(\chi) = e^{-\left[\frac{2a_1\chi^{(2-M)}}{\mu(M-2)}\right]} & \left\{ c_1 + \frac{1}{M(M-2)} \left[2^{\frac{M}{M-2}} N \chi^2 \Gamma\left(-\frac{2}{M-2}, \right. \right. \right. \\ & \left. \left. \left. -\frac{2a_1\chi^{(2-M)}}{\mu(2-M)}\right) \right] \left(\frac{a_1\chi^{(2-M)}}{\mu(2-M)} \right)^{\left(\frac{2}{M-2}\right)} \right\}. \end{aligned} \quad (34)$$

where c_1 is the arbitrary integration constant, and $\Gamma(a, z) = \int_z^\infty t^{(a-1)} e^{-t} dt$, is the incomplete gamma function.

Obviously, the case studied earlier in this work is obtained from the above when one chooses $M = 2$. On the other hand, we can get simpler solutions for other particular values of the parameter M as, for instance $M = 4$, whose solution can be written as

$$\sigma_4(\chi) = \frac{N}{4} \chi^2 + e^{-\frac{a_1}{\mu\chi^2}} \left[c_1 + \frac{N a_1}{4\mu} Ei\left(\frac{a_1}{\mu\chi^2}\right) \right], \quad (35)$$

where $Ei(z) \equiv -\int_{-z}^\infty \frac{e^{-t}}{t} dt$, is the exponential integral function. It can be seen from Fig.3 that, apart from a small region close to the origin, it is

asymptotically similar to that of the case with $M = 2$, which was discussed in some detail above in the text. This expression does not have any kind of singularity and approaches to zero when the field χ does the same. Notwithstanding, the last part of the analysis of the kinks needs to be done through evaluation of the equation

$$\frac{d\chi}{dx} = \mu \chi^{(M-1)} (\pm)^N \left[-\left(\frac{a_0}{a_1}\right) + \sigma_M(\chi) \right]^{\frac{N}{2}}, \quad (36)$$

which is not easy to be done analytically, so that one needs to make use of numerical techniques. We intend to perform this analysis in a future work, looking for new interesting features.

Acknowledgments: The author is grateful to CNPq for partial financial support, to the Professor D. Bazeia for introducing him to this matter, and to the referee for the very pertinent and constructive criticisms and suggestions. This work has been finished during a visit within the Associate Scheme of the Abdus Salam ICTP.

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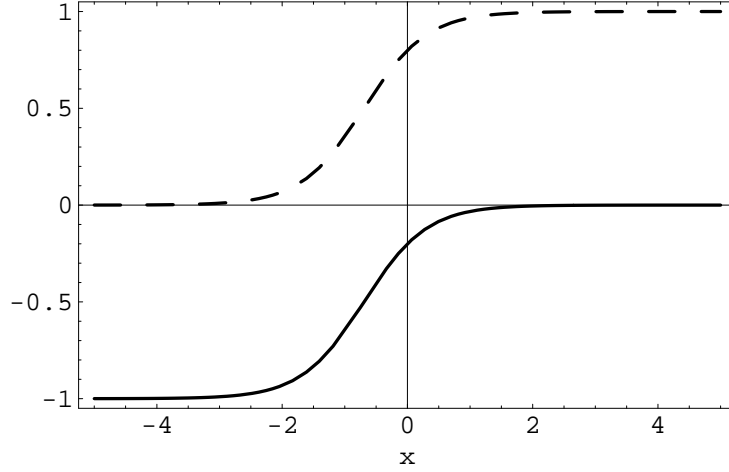


Figure 1: Typical type B kink profile (for $\lambda = \mu$). The dotted line corresponds to the field $\chi_+(x)$ and the solid line to the field $\phi_+(x)$. Both were calculated for $c_0 = -2$.

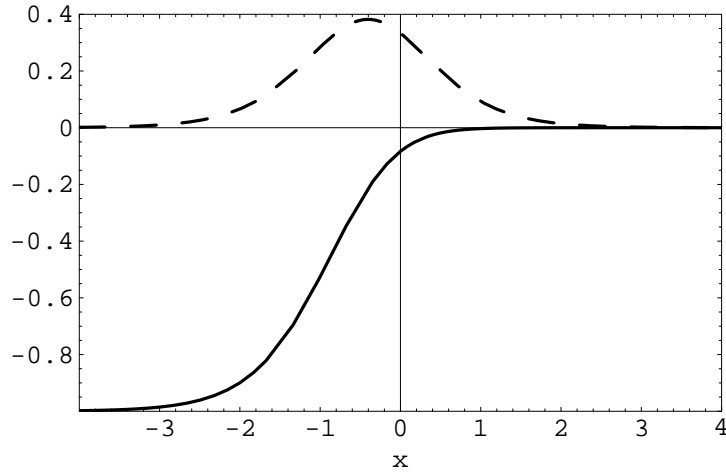


Figure 2: Typical type A kink profile (for $\lambda = \mu$). The dotted line corresponds to the field $\chi_+(x)$ and the solid line to the field $\phi_+(x)$. Both were calculated for $c_0 = -3$.

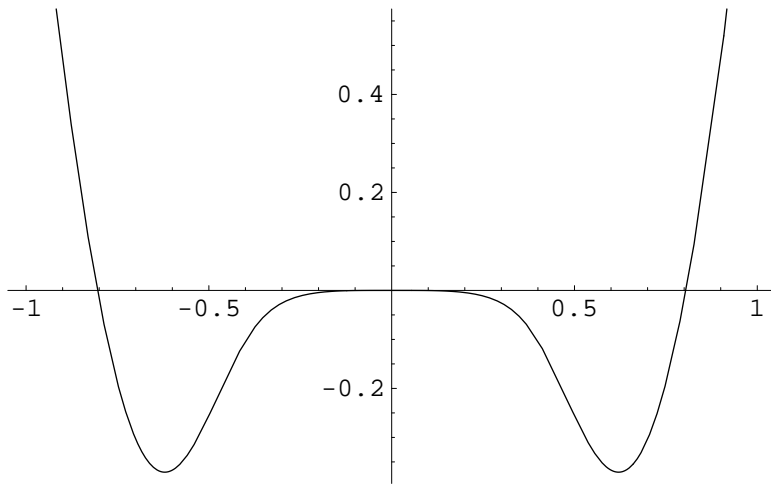


Figure 3: The function σ , for $M = 4$ as a function of the field χ , as defined in Eq. (35).